UNIVERSITY OF JOHANNESBURG

DEPARTMENT OF ECONOMICS AND ECONOMETRICS

UNIVERSITY OF JOHANNESBURG

Aukland Park Kingsway Campus

Assessor: Ms Q M Mabe

Internal Examiner: Prof

Final Exam : QUANTITATIVE ECONOMICS QTE3BB3

24 November 2016

TIME: 2h30 Hours

TOTAL MARKS: 80

Instructions:

- Answer all the questions
- Write neatly and legibly
- Justify all your steps with mathematical theory
- Use pen, not pencil

Question 1

1. The equation $y'' - y = e^x$ has a general solution of the form $\left[\text{ 2 marks }\right]$

   (a) $y = Ax + C$

   (b) $y = Ax^2 + Bx + C$

   (c) $y = Ae^{xr_1} + Be^{xr_2} + Ce^x$

   (d) $y = Ae^{xr_1} + Be^{xr_2}$

   (e) $y = Ce^x$

2. Suppose **y*** is a steady state of the first order system of differential equation **y'=F(y*)** on R^n. If each eigenvalue of the Jacobian matrix $DF(\mathbf{y*})$ of F at **y*** is negative or has negative real part then

   (a) **y*** diverges

   (b) **y*** is an asymptotically stable steady state of **y'=F(y)**

   (c) **y*** is an unstable steady state of **y'=F(y)**

   (d) the system **y'=F(y)** never reaches equilibrium

   (e) none of the above

3. A basis for the null space of $A = \begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix}$
   is $\left[\text{ 2 marks }\right]$

   (a) $\begin{pmatrix} 1 \\ \frac{1}{4} \end{pmatrix}$
   or any multiple of it

   (b) $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ -\frac{1}{7} \end{pmatrix} \right\}$

   (c) $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

   (d) Any set of orthogonal vectors

   (e) none of the above

4. Given that A is an $n \times n$ matrix, A is said to be diagonalisable if $\left[\text{2 marks}\right]$

   (a) It has a set of n linearly dependent eigen-vectors

   (b) If there exist a matrix P such that $PAP^{-1} \neq D$

   (c) If there are repeated eigen-values and the set of eigen-vectors is given as $S = \{V_1, V_2 \cdots\cdots V_m\}$ where $m < n$

   (d) If there are distinct eigen-values and the set of eigen-vectors is given as $S = \{V_1, V_2 \cdots\cdots V_m\}$ where $m = n$

   (e) If the characteristic polynomial has complex roots

5. The first integral of the system of differential equations $x' = y$ and $y' = -x$ is $\left[\text{ 2 marks}\right]$

   (a) $F(x, y) = xy$

   (b) $F(x, y) = x^2 + y$

(c) $F(x, y) = x^2 + y^2$

(d) $F(x, y) = x + y$

(e) $F(x, y) = x - y$

TOTAL MARKS $\left[10 \text{ marks}\right]$

QUESTION 2

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (2x + y - 2z, 2x + 3y - 4z, x + y - z)$
   (a) Find the matrix A that represents T $\left[\text{ 1 Mark }\right]$
   (b) Find the charecteristic polynomial of A and the eigenvalues $\left[\text{ 4 Marks }\right]$
   (c) Find the linearly independent eigen-vectors for each eigenvalue of A $\left[\text{ 4 Marks }\right]$
   (d) Is T diagonalisable? State a reason to your answer. $\left[\text{ 3 Marks }\right]$
2. Prove that the diagonal entries of a diagonal matrix D are eigenvalues of D. $\left[\text{ 4 Marks }\right]$
3. Prove that if A is a 3×3 matrix with an eigenvalue r^* of multiplicity three and with three independent eigenvectors, then A must equal the diagonal matrix r^*I. $\left[\text{ 4 Marks }\right]$

   Total Marks : 20

QUESTION 3

1. Given that $B = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$

   (a) Find all eigen-values of B [2 marks]

   (b) Find the orthogonal matrix which diagonalises B. [6 marks]

2. Show that the general solution of $y' = t - y$ is $y = ke^{-t} + t - 1$, and find the solution that satisfies $y(1) = 1$. [8 marks]

3. Show that $y = x - x^{-1}$ is a solution of the differential equation $xy' + y = 2x$

[
4 marks]
Total Marks [20 marks]

QUESTION 4

1. Sketch the direction field of $y' = x + y^2$ [4 marks]
2. Solve the differential equation $\frac{dp}{dt} = t^2 p - p + t^2 - 1$ [4 marks]
3. Find the general solution of the second order differential equation, $y'' - 2y' - 3y = cos2x$. [6 marks]
4. Solve the initial value problem for $2y'' + 5y' + 3y = 0$ $y(0) = 3$,$y'(0) = -4$. [6 marks]

Total Marks [20 marks]

QUESTION 5

1. Find the steady state of the system for $\mathbb{R} > 0$ and establish its stability
   $x' = 2xy - 2y^2$
   $y' = x - y^2 + 2$ $\left[5 \text{ marks}\right]$

2. A vector field on $\mathbb{R}^2$ is defined by $F(x, y) = (-y, x)$. Describe F by sketching some of its vectors.$\left[5 \text{ marks}\right]$
   Total Marks $\left[\ 10 \text{ marks}\ \right]$